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Dynamical Interpretation for the Quantum-Measurement Projection Postulate

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An apparatus model with discrete momentum space suitable for the exact solution of the problem is considered. The special Hamiltonian of its interaction with the object system under consideration is chosen. In this simple case it is easy to illustrate how difficulties in constructing the dynamical interpretation of selective collapse can be overcome without any limiting procedure. For this purpose one can apply either averaging with respect to a nonquantum parameter or reduce the algebra of joint-system operators (i.e., pass from an algebra $\mathscr{A}$ of operators to a subalgebra $\mathscr{A}_0$). The latter procedure implies averaging with respect to apparatus quantum variables not belonging to $\mathscr{A}_0$.

1. INTRODUCTION

In this paper we consider the dynamical interpretation of the selective collapse in the one-dimensional case when the momentum of the apparatus has a discrete spectrum of eigenvalues. This simplifies the problem of dynamical corroboration of the von Neumann projection postulate. The idea to consider the case when one of two main conjugate dynamic variables (momentum or coordinate) is discrete and to take the apparatus state commuting with discrete variable comes from Belavkin (1994).

The approach to the problem of the selective collapse interpretation is quite standard and is well known from the time of von Neumann (1955). The collapse of the quantum object state which takes place during measurement of an object variable X with discrete spectrum is interpreted with the help of interaction between object S and apparatus A (the latter being in the quasiclassical state) and with the help of the subsequent classical-like measurement

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of some apparatus variable Y. In our case Y depends on quantum momentum $\hat{p}$. Moreover, in this case the evolution operator can exactly realize the transformation of the product wave function

$$|\varphi\rangle \otimes |y_0\rangle = \sum_j c_j |x_j\rangle \otimes |y_0\rangle \tag{1}$$

for the joint system into the correlated one

$$\sum_j c_j |x_j\rangle \otimes |y_j\rangle \tag{2}$$

This transformation was proposed by von Neumann for the measurement interpretation. Here $|x_j\rangle$ are the eigenfunctions of X, and $|y_j\rangle$ are the eigenfunctions of Y. In contrast to the von Neumann theory, use of the mixed apparatus state or, to be exact, the quasiclassical state is desirable for us because eigenvalues $\{y_j\}$ of Y can only be distinguished from one another macroscopically in a quasiclassical state, the appropriate measured operator Y being chosen. Moreover, we use an averaging procedure of the apparatus state (Stratonovich, 1995). This procedure helps to overcome the difficulties connected with the dynamic interpretation of the collapse; it makes the apparatus state compatible with Y.

Our goal is to interpret the selective collapse

$$\rho_S \to \frac{1}{w_l} E_l \rho_S E_l \tag{3}$$

($w_l = \mathrm{Tr}_S\, \rho_S E_l$) of the density matrix of the quantum object S. According to the projection postulate it takes place when the result x_l of measurement of the operator $X = \sum_j x_j E_j$ becomes known. Here E_j are the orthogonal projectors ($E_j E_k = E_j \delta_{jk}$, $\sum_j E_j = I_S$).

Our treatment is restricted to the following assumptions.

(i) The coordinate space of the apparatus model is finite, namely it is of length L and is curved into itself (like a circumference), the coordinate space being, say, the interval $[-L/2, L/2]$. This means that the shift $q \to q + a$ gives $q + a - L$ if $L/2 < q + a < 3L/2$ and $q + a + L$ if $-L/2 > q + a > -3L/2$. The pointer on a fixed axis (for which $q = \varphi$ is the angle, $L = 2\pi$) or a box with periodic boundary conditions may serve as examples. For an arbitrary L the apparatus momentum has eigenvalues $p_k = 2\pi\hbar k/L$.

(ii) The initial apparatus density matrix ρ_A is compatible with momentum $\hat{p}$, i.e., is diagonal in the momentum representation

$$\rho_{kl} \overset{\text{def}}{=} \langle p_k | \rho_A | p_l \rangle = w_k^0 \delta_{kl} \tag{4}$$

In addition we suppose that

$$w_k^0 = 0 \qquad \text{at} \quad |k| > m \tag{5}$$

This compatible density matrix is only possible because of the discrete character of the momentum spectrum. In fact, its continuous variant

$$\langle p' | \rho_A | p \rangle = w^0(p)\delta(p' - p) \quad (6)$$

is impossible because this operator has infinite trace [if $w^0(p)$ is not equal to zero everywhere].

(iii) The interaction Hamiltonian is of the form

$$H_{\text{int}}(t) = -B \otimes (\gamma \hat{q} + \lambda I_A)\delta(t) \quad (7)$$

where γ, λ are interaction constants and I_A is the apparatus identity operator. Of course, the presence of a delta function on the right-hand side of (7) makes the process of interaction somewhat unrealistic. This delta-function type of interaction was applied in Belavkin (1995) in the recurrent variant for realizing continuous observation.

The operator

$$B = f(X) = \sum_j b_j E_j = \sum_j (x_j) E_j \quad (8)$$

enters the right-hand side of (7), the function f being chosen in such a way that all eigenvalues b_j of B are multiples of the quantity $a > 0$:

$$b_j = n_j a \quad (9)$$

Here n_j are integers that increase with increasing j. The transformation $b_j = f(x_j)$ is supposed to be nondegenerate. The necessity of (9) will be clear later.

To obtain the collapse (3) of the object system state, the measurement of the variable Y depending on the apparatus momentum will be made. The matrix density (4) is very convenient for measuring Y because it commutes with $\hat{p}$ and therefore with $Y(\hat{p})$.

In the general case the selective quantum collapse

$$\rho_A \to \frac{1}{w_l'} P_l \rho_A P_l \quad (10)$$

of the apparatus state takes place after measurement of $Y = \Sigma_j\, y_j P_j$ if the measurement result y_l becomes known. Here P_j are eigenorthoprojectors of Y ($\Sigma_j\, P_j = I_A$) and $w_l' = \text{Tr}_A\, P_l \rho_A$. Averaging $\Sigma_l\, w_l' \tilde{\rho}_l$, one finds that the *a posteriori* matrix densities

$$\tilde{\rho}_l = \frac{1}{w_l'} P_l \rho_A P_l \quad (11)$$

do not give the *a priori* matrix ρ_A in the general case. This means that the condition of consistency

$$\sum_l w_l' \tilde{\rho}_l = \rho_A \qquad \text{or} \qquad \sum_l P_l \rho_A P_l = \rho_A \tag{12}$$

is not obliged to be met. In our case the projectors P_j defined by

$$P_j = \vartheta_j(\hat{p}) \tag{13}$$

commute with ρ_A and therefore the consistency condition (12) is met. The functions $\vartheta_j(\xi)$ are defined by (52).

As was pointed out in Stratonovich (1995), the quasiclassical collapse

$$\rho_A \to \frac{1}{w_l'} \rho_A * P_l \tag{14}$$

obviously meeting the consistency condition, can be applied in some cases. Here the operation $*$ is defined with the help of the Wigner transformation (27), (29) denoted by $\mathcal{W}$. To be exact, in our case

$$A * B = L\mathcal{W}^{-1}\{\mathcal{W}[A]\mathcal{W}[B]\} \tag{15}$$

For projectors (13) we have

$$L\mathcal{W}[\vartheta_l(\hat{p})] = \vartheta_l(p_j) \tag{16}$$

and (14) is equivalent to

$$\rho_A \to (w_l')^{-1}\mathcal{W}^{-1}[\mathcal{W}[\rho_A]\vartheta_l(p_j)] \tag{17}$$

or (if we apply $\mathcal{W}$ to both sides of the last formula)

$$w(q, p_j) \to \frac{1}{w_l'} w(q, p_j)\vartheta_l(p_j) \tag{18}$$

This is nothing else but the transition to the conditional distribution, which is a well-known nonquantum procedure. Using (32), one can easily see that collapse (14), (18) is exactly equivalent to (10) in the simple case (4). Because of this fact and because the condition (12) is met in our case, we call the measurement of $Y = \Sigma\, y_k\vartheta_k(\hat{p})$ classical-like.

2. THE INITIAL APPARATUS STATE IN OTHER REPRESENTATIONS

Eigenfunctions of momentum $\hat{p}$ corresponding to the eigenvalues $p_k = 2\pi\hbar k/L$ are

$$\psi_k(q) = L^{-1/2} \exp(ip_k q/\hbar) \tag{19}$$

(the coordinate representation). Using expression on the right-hand side taken at various k, we readily can write down the matrix elements

$$V_{qk} = L^{-1/2} \exp(2\pi ikq/L) \tag{20}$$

of the unitary operator V transforming the $\hat{p}$-representation to the $\hat{q}$-representation and vice versa. Thus the $\hat{q}$-representation of the density matrix is

$$\rho(q', q) \stackrel{\text{def}}{=} \langle q' | \rho_A | q \rangle = \sum_{kl} V_{q'k} \rho_{kl} V^{\dagger}_{lq} \tag{21}$$

or, due to (4) and (20),

$$\rho(q', q) = L^{-1} \sum_k \exp[2\pi i(q' - q)k/L]\, w_k^0 \tag{22}$$

Therefore the coordinate probability density $w^0(q) = \rho(q, q)$ is uniform,

$$w^0(q) = 1/L \tag{23}$$

Hence we find the coordinate mean $\langle q \rangle = 0$ and mean square

$$\sigma_q^2 \stackrel{\text{def}}{=} \langle q^2 \rangle = \frac{1}{L} \int_{-L/2}^{L/2} q^2\, dq = \frac{L^2}{12} \tag{24}$$

On the other hand, the momentum mean square is

$$\sigma_p^2 = \sum_{k=-m}^{m} p_k^2 w_k^0 = 8\pi^2 \hbar^2 L^{-2} \sum_{k=0}^{m} k^2 w_k^0 \tag{25}$$

if $w_{-k}^0 = w_k^0$. According to (24), (25), we have

$$\sigma_q^2 \sigma_p^2 = \frac{2}{3} \pi^2 \hbar^2 \sum_{k=0}^{m} k^2 w_k^0 \tag{26}$$

It should be noted that we get $\sigma_q \sigma_p = 0$ from (26) if $m = 0$, i.e., if $w_k^0 = \delta_{k0}$. This equation is very unusual since it violates the Heisenberg uncertainty relation $\sigma_q \sigma_p \geq \hbar/2$. The possibility of this paradox is substantiated in the Appendix.

When $\sigma_q \sigma_p >> \hbar$, the apparatus is in a quasiclassical state. We will suppose that this inequality is valid because the direct macroscopic observation of a physical quantity is possible only in this case. Owing to (5) and the normalization condition $\Sigma_k w_k^0 = 1$, the inequality $m >> 1$ is a necessary condition for $\sigma_q \sigma_p >> \hbar$. For many kinds of distribution, e.g., for the uniform one, $m >> 1$ is also a sufficient condition of a quasiclassical state.

Another representation of the apparatus state is the Wigner distribution, which in our case takes the form

$$w(q, p_j) = \frac{1}{L}\int_{-L/2}^{L/2} \exp\left(-\frac{i}{\hbar} up_j\right) \rho\left(q + \frac{u}{2}, q - \frac{u}{2}\right) du$$

$$= \frac{1}{L}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \exp\left[\frac{i}{\hbar} q(p_k - p_l)\right] \Delta\left(\frac{k+l}{2} - j\right)\rho_{kl} \qquad (27)$$

Here $\Delta(\eta) = \int_{-1/2}^{1/2} \exp(2\pi i\eta\nu)\, d\nu$, i.e.,

$$\Delta(\eta) = \frac{\sin(\pi\eta)}{\pi\eta} = \begin{cases} \delta_{n0} & \text{at} \quad \eta = \eta \\ (-1)^n \pi^{-1}/(n + 1/2) & \text{at} \quad \eta = n + 1/2 \end{cases} \qquad (28)$$

(n is integer). We denote the transformation (27) by $\mathcal{W}$:

$$\mathcal{W}[\rho_A] = w(q, p_j) \qquad (29)$$

It is easy to check that $w(q, p_j)$ has the properties

$$\sum_j w(q, p_j) = \rho(q, q), \qquad \int w(q, p_j)\, dq = \rho_{jj} = w_j^0 \qquad (30)$$

usual for the Wigner distribution. Moreover the formula

$$\mathrm{Tr}_A\, G\rho_A = L \sum_{j=-\infty}^{\infty} \int_{-L/2}^{L/2} \mathcal{W}[G]\mathcal{W}[\rho_A]\, dq \qquad (31)$$

is valid. For the special matrix density (4) we get

$$w(q, p_j) = w_j^0/L \qquad (32)$$

3. INTERACTION BETWEEN THE OBJECT SYSTEM S AND APPARATUS

Let H_S be a Hamiltonian acting in the Hilbert space $\mathcal{H}_S$ of the object system S. The apparatus Hamiltonian H_A is an operator acting in $\mathcal{A}_A$. An interaction between S and A that lasts a very short time from $t = -\epsilon$ to $t = \epsilon > 0$ is described by the interaction Hamiltonian (7) acting in $\mathcal{H}_S \otimes \mathcal{H}_A$, B being the S-system operator with discrete eigenvalues (9). Its measurement or—what is equivalent—measurement of X is to be interpreted. The total Hamiltonian assumes the form

$$H(t) = H_S \otimes I_A + I_S \otimes H_A - \gamma B \otimes \hat{q}\delta(t) - \lambda B \otimes I_A\delta(t) \qquad (33)$$

The state of the joint system S + A at the initial instant $t_0 = -\epsilon$ is given by the density matrix

$$\rho(-\epsilon) = \rho_S \otimes \rho_A \tag{34}$$

In the Schrödinger picture the density matrix depends on time as

$$\rho(t_1) = U(t_1, t_0)\rho(t_0)U^\dagger(t_1, t_0) \tag{35}$$

where the evolution operator U is given by

$$U(t_1, t_0) = \mathcal{T} \exp\left[-\frac{i}{\hbar} \int_{t_0}^{t_1} H(t)\, dt \right] \tag{36}$$

Here $\mathcal{T}$ denotes the time ordering of operators $H(t)$, namely the greater t is, the more to the left $H(t)$ stands. We choose $t_1 = \epsilon$, where ϵ is a very small positive number. Then (35) gives

$$\rho(\epsilon) = \exp\left[\frac{i}{\hbar} B \otimes (\gamma\hat{q} + \lambda I_A) \right] (\rho_S \otimes \rho_A) \exp\left[-\frac{i}{\hbar} B \otimes (\gamma\hat{q} + \lambda I_A) \right] \tag{37}$$

owing to (33), (34), and (36). We will use the orthogonal projectors $\{E_j\}$ corresponding to the operator $B = \Sigma_j\, b_j E_j$. As is well known, for them

$$I_S = \sum_j E_j \tag{38}$$

By virtue of (38) we can take $\Sigma_i\, E_i\rho_S\, \Sigma_j E_j$ instead of ρ_S in (37) and obtain

$$\rho(\epsilon) = \sum_{ij} \exp\left[\frac{i}{\hbar} B \otimes (\gamma\hat{q} + \lambda I_A) \right] (E_i \rho_S E_j \otimes \rho_A)$$
$$\times \exp\left[-\frac{i}{\hbar} B \otimes (\gamma\hat{q} + \lambda I_A) \right] \tag{39}$$

But $BE_i = b_i E_i$, $E_j B = E_j b_j$, and $g(B \otimes D)(E_j \otimes \rho_A) = E_j \otimes (g(b_j D)\rho_A)$ for an arbitrary c-function g. Therefore (39) yields

$$\rho(\epsilon) = \sum_{ij} E_i \rho_S E_j \otimes \exp\left[\frac{i}{\hbar} b_i(\gamma\hat{q} + \lambda I_A) \right] \rho_A \exp\left[-\frac{i}{\hbar} b_j(\gamma\hat{q} + \lambda I_A) \right] \tag{40}$$

Now we use formula (9) and let

$$a\gamma = 2\pi\hbar(2m + 1)/L \tag{41}$$

Then in the apparatus coordinate representation

$$\langle q' | \rho(\epsilon) | q \rangle = \sum_{ij} E_i \rho_S E_j \exp[i(n_i - n_j)\chi]$$
$$\times \exp[2\pi i(2m + 1)(n_i q' - n_j q)L^{-1}]\, \rho(q', q) \tag{42}$$

with $\chi = a\lambda/\hbar$. Substituting (22) into the right-hand side and passing to the p-representation, we get

$$\langle p_r | \rho(\epsilon) | p_s \rangle = \sum_{ij} E_i \rho_S E_j e^{i(n_i - n_j)\chi} w^0_{r-(2m+1)n_i} \delta_{r-s-(2m+1)(n_i-n_j)} \tag{43}$$

($\delta_{kl} = \delta_{k-l}$). This result implies the following Wigner transform:

$$\mathcal{W}[\rho(\epsilon)]_{q,p_k} = \sum_{ij} E_i \rho_S E_j \exp[i(n_i - n_j)\chi] \exp\left[2\pi i(2m + 1)(n_i + n_j)\frac{q}{L}\right] \times w(q, p_k - \tfrac{1}{2}(p_{(2m+1)n_i} + p_{(2m+1)n_j})) \tag{44}$$

if all $n_i + n_j$ are even.

4. THE APPARATUS PHYSICAL QUANTITY THAT SHOULD BE MEASURED

Let us consider the expression

$$R(p_r) \overset{\text{def}}{=} \langle p_r | \rho(\epsilon) | p_r \rangle \tag{45}$$

which in our case, due to (43), assumes the form

$$R(p_r) = \sum_j E_j \rho_S E_j w^0_{r-(2m+1)n_j} \tag{46}$$

It is an operator in $\mathcal{H}_S$ and simultaneously the distribution of momentum p_j. We see that correlation between values b_j of B and those of $\hat{p}$ exists in (46). In fact, the density matrix

$$\tilde{\rho}_j = \frac{1}{w_j} E_j \rho_S E_j \tag{47}$$

in which B has definite value b_j, enters the same term $w_j \tilde{\rho}_j w^0_{r-(2m+1)n_j}$ of the sum (46) as the distribution $w^0_{r-(2m+1)n_j}$, which differs from 0 in the range

$$-2\pi\hbar m/L \leq p_r - 2\pi\hbar(2m + 1)n_j/L \leq 2\pi\hbar m/L \tag{48}$$

[according to (5)], i.e.,

$$2\pi\hbar[(2m + 1)n_j - m]/L \geq p_r \leq 2\pi\hbar[(2m + 1)n_j + m]/L \tag{49}$$

Therefore determining the range to which the momentum belongs signifies determining the value of B and X. Let us denote the range (49) by S_j. Thus

$$w^0_{r-(2m+1)n_j} = \begin{cases} w^0_{r-(2m+1)n_j} & \text{at} \quad p_r \in S_j \\ 0 & \text{at} \quad p_r \notin S_j \end{cases} \tag{50}$$

Various ranges never overlap because $n_{j+1} - n_j \geq 1$. Let us introduce the enlarged nonoverlapping ranges $\tilde{S}_j$ such that each $\tilde{S}_j$ includes S_j and so that the sum $\Sigma_j \tilde{S}_j$ is equal to the set of all p_j, $j = 0, \pm 1, \pm 2, \ldots$. This enlarging can be made in various ways. For example, we can consider the points

$$s_j = 2\pi\hbar L^{-1}\left[(2m+1)\frac{n_j + n_{j+1}}{2}\right]_{\mathrm{IP}} \tag{51}$$

(the subscript IP means the integral part) lying approximately halfway between S_j and S_{j+1} and define $\tilde{S}_j$ as the range $s_{j-1} < p_k \leq s_j$. Now we define the function

$$\vartheta_j(p_k) = \begin{cases} 1 & \text{at} \quad p_k \in \tilde{S}_j \\ 0 & \text{otherwise} \end{cases} \tag{52}$$

From (50), (52) and since S_j is the subset of $\tilde{S}_j$, we have

$$w^0_{k-(2m+1)n_i}\vartheta_j(p_k) = w^0_{k-(2m+1)n_i}\delta_{ij} \tag{53}$$

Let the measured apparatus operator be

$$Y(\hat{p}) = \sum_j p_{(2m+1)n_j}\vartheta_j(\hat{p}) \tag{54}$$

($p_{(2m+1)n_j}$ being the central point of S_j), or

$$Y = \sum_j j\vartheta_j(\hat{p}) \tag{55}$$

Equation (54) corresponds to inexact measurement of $\hat{p}$; the latter one means that the number j of the range to which p belongs is measured. Note that we may set $Y = \Sigma_j x_j \vartheta_j(\hat{p})$; then we will have $\langle [X \otimes I_{\mathrm{A}} - I_{\mathrm{S}} \otimes Y]^2 \rangle = 0$ as follows from (62), (46), (53).

5. SELECTIVE COLLAPSE OF THE S-SYSTEM STATE AS A RESULT OF MEASURING APPARATUS VARIABLE Y

Now if we measure the physical quantity (54) or (55) and p proves to belong to $\tilde{S}_l$, the collapse

$$\rho(\epsilon) \to \frac{1}{w'_l}[I_{\mathrm{S}} \otimes \vartheta_l(\hat{p})]\rho(\epsilon)[I_{\mathrm{S}} \otimes \vartheta_l(\hat{p})] \tag{56}$$

(with $w'_l = \mathrm{Tr}[I_{\mathrm{S}} \otimes \vartheta_l(\hat{p})]\rho(\epsilon)[I_{\mathrm{S}} \otimes \vartheta_l(\hat{p})]$) takes the form

$$\langle p_r | \rho(\epsilon) | p_s \rangle \to \frac{1}{w'_l}\vartheta_l(p_r)\langle p_r | \rho(\epsilon) | p_s \rangle \vartheta_l(p_s)$$

$$= \frac{1}{w'_l} E_l \rho_{\mathrm{S}} E_l w^0_{s-(2m+1)n_l}\delta_{rs} \tag{57}$$

owing to (43), (53). In fact, applying (53), we have

$$\begin{aligned}\vartheta_l(p_r)w^0_{r-(2m+1)n_i}\delta_{r-s-(2m+1)(n_i-n_j)} \\ = w^0_{r-(2m+1)n_i}\delta_{r-s-(2m+1)(n_i-n_j)}\delta_{il} \\ = w^0_{s-(2m+1)n_j}\delta_{r-s-(2m+1)(n_i-n_j)}\delta_{il}\end{aligned} \tag{58}$$

and

$$w^0_{s-(2m+1)n_j}\vartheta_l(p_s) = w^0_{s-(2m+1)(n_l}\delta_{jl} \tag{59}$$

This leads to (57). Formula (57) means that the *a posteriori* state of quantum object S is $E_l\rho_S E_l/w_l' = E_l\rho_S E_l/w_l$.

However, the objection arises that it is incorrect to interpret the quantum collapse $\rho_S \to E_l\rho_S E_l/w_l$ by another quantum collapse, namely by (56). In fact, the matrix (43) does not commute with $I_S \otimes \hat{p}$ and $I_S \otimes Y$ and therefore a consistency condition of the type (12) is violated. This condition would had been met for collapse

$$\rho(\epsilon) \to \frac{1}{w_l'}\rho(\epsilon) * \vartheta_l(\hat{p}) \tag{60}$$

but now (60) is not justified since it contradicts the collapse (56).

To overcome the above difficulty, the averaging with respect to some quantum or nonquantum variables should be done. There are several lines of action and reasoning.

1. We suppose that the nonquantum parameter χ entering the right-hand side of (43) is random and uniformly distributed on the interval $-\pi < \chi \leq \pi$. Then averaging the right-hand side of (43) with respect to χ leads to

$$\overline{\langle p_r | \rho(\epsilon) | p_s\rangle} = \sum_j E_j\rho_S E_j w^0_{r-(2m+1)n_j}\delta_{rs} \tag{61}$$

because the mean value of $\exp[i(n_i - n_j)\chi]$ is δ_{ij}. The matrix density (61) commutes with $I_S \otimes \hat{p}$ and $I_S \otimes Y(\hat{p})$. Therefore the measurement of Y is classical-like (see Section 1) and both the quantum collapse (56) and the classical one (60) may now be applied to (61). This gives the resulting *a posteriori* state $E_l\rho_S E_l w^0_{r-(2m+1)n_l}\delta_{rs}/w_l'$. Averaging with respect to the apparatus parameter was used in Machida and Namiki (1980) for explaining the nonselective collapse.

2. Another possibility is averaging with respect to some quantum variables of the apparatus. We can restrict the operator algebra in which we are interested. Let us only consider operator subalgebra $\mathcal{A}_0$ generated by all operators of the S-system (i.e., operators of the type $D \otimes I_A$) and by the operator $I_S \otimes \hat{p}$. The analogous type of operator subalgebra (with coordinate taken instead of momentum) was considered in Belavkin (1994). Namely the algebra of all operators commuting with $Q = \kappa I \otimes \hat{q}$ was applied there for securing the consistency condition in defining nondemolition observation continuous in time, the operator Y having both discrete and continuous spectrum. Earlier Araki (1980) used a special subalgebra of operators for obtaining nonselective collapse in the limit $t \to \infty$ for a particular choice of interaction.

The state functional (functional of mean values) for operators belonging to our subalgebra $\mathcal{A}_0$ is defined with the help of the operator (45):

$$\langle G \rangle = \sum_k \mathrm{Tr}_S\, R(p_k) \langle p_k | G | p_k \rangle \tag{62}$$

When we only consider operators from the subalgebra $\mathcal{A}_0$ and use $R(p_k)$, the classical selective collapse

$$R(p_k) \to \frac{1}{w_l'} R(p_k) \vartheta_l(p_k) \tag{63}$$

analogous to transition to the conditional probability distribution takes place provided that the result of the measurement becomes known. According to (46), (53) this means the transformation

$$R(p_k) \to \frac{1}{w_l'} E_l \rho_S E_l w^0_{k-(2m+1)n_l} \tag{64}$$

Summation with respect to the apparatus momentum gives the *a posteriori* state $E_l \rho_S E_l / w_l'$ of the quantum object.

3. Suppose now that the quantum system interacts with two systems A and C, C being another copy of the A-system considered earlier. Let it be in the same initial state. Then A + C constitute a new complex apparatus. Averaging with respect to the C-system variables, i.e., considering subalgebra $\mathcal{A}_0$ of operators of the type $D \otimes I_C$ (D being an operator in $\mathcal{H}_S \otimes \mathcal{H}_A$) will solve the problem. For operators $\tilde{D} = D \otimes I_C$ from $\mathcal{A}_0$ the functional of mean values is $\langle \tilde{D} \rangle = \mathrm{Tr}_{S+A}\, D\rho_{S+A}$ with $\rho_{S+A} = \mathrm{Tr}_C\, \rho$.

Now the total Hamiltonian takes the form

$$H(t) = H_S'' + H_A'' + H_C'' - \gamma B''(\hat{q}'' + Q'')\delta(t) \tag{65}$$

where $H_S'' = H_S \otimes I_A \otimes I_C$, $B'' = B \otimes I_A \otimes I_C$, $\hat{q}'' = I_S \otimes \hat{q} \otimes I_C = I_S \otimes \hat{q}'$, $Q'' = I_S \otimes I_A \otimes Q = I_S \otimes Q'$, and so on, Q being the coordinate of C, i.e., the operator in $\mathcal{H}_C$. Naturally the matrix

$$\rho = \rho_S \otimes \rho_A \otimes \rho_C \tag{66}$$

serves as the initial density matrix. In this case we have

$$\rho(\epsilon) = \sum_{ij} E_i\rho_S E_j \otimes \exp\left[\frac{i}{\hbar}\gamma b_i(\hat{q}' + Q')\right](\rho_A \otimes \rho_C)$$
$$\times \exp\left[-\frac{i}{\hbar}\gamma b_j(\hat{q}' + Q')\right] \tag{67}$$

instead of (40). Since $\hat{q}'$ commutes with Q' and $I_A \otimes \rho_C$ and Q' commutes with $\rho_A \otimes I_C$, this formula can be written as

$$\rho(\epsilon) = \sum_{ij} E_i\rho_S E_j \otimes \exp(i\gamma b_i\hat{q}/\hbar)\, \rho_A \exp(-i\gamma b_j\hat{q}/\hbar)$$
$$\otimes \exp(i\gamma b_i Q/\hbar)\, \rho_C \exp(-i\gamma b_j Q/\hbar) \tag{68}$$

If we write the matrices $r_{ij} = \exp(i\gamma b_i Q/\hbar)\rho_C \exp(-i\gamma b_j Q/\hbar)$ in the coordinate representation, we have

$$r_{ij}(Q', Q) = \exp[i\hbar^{-1}\gamma a(n_i Q' - n_j Q)]\, \rho_C(Q', Q) \tag{69}$$

where, according to (5),

$$\rho_C(Q', Q) = L^{-1} \sum_{k=-m}^{m} \exp[2\pi i(Q' - Q)k/L]\, w_k^0 \tag{70}$$

[(70) is analogous to (22)]. From (69), (70) we see that setting $\gamma a = 2\pi\hbar N/L$ (N is an integer) and taking the partial trace Tr_C with respect to the C-system (i.e., integrating with respect to $Q' = Q$) gives

$$\mathrm{Tr}_C\, r_{ij} = \delta_{ij} \tag{71}$$

Therefore we get from (68)

$$\mathrm{Tr}_C\, \rho(\epsilon) = \sum_j E_j\rho_S E_j \otimes [\exp(i\gamma a n_j\hat{q}/\hbar)\, \rho_A \exp(-i\gamma a n_j \hat{q}/\hbar)] \tag{72}$$

and

$$\langle p_k | \mathrm{Tr}_C\, \rho(\epsilon) | p_l\rangle = \sum_j E_j\rho_S E_j w^0_{k-(2m+1)n_j}\delta_{kl} \tag{73}$$

for $N = 2m + 1$. Thus, averaging with respect to all C-system quantum variables gives the same result as averaging in the nonquantum random parameter χ.

In the first and third ways of reasoning we obtain the *a posteriori* combined system state $\tilde{\rho}_l^{(S)} \otimes \tilde{\rho}_l^{(A)}$, where

$$\tilde{\rho}_l^{(S)} = E_l\rho_S E_l/w_l', \qquad \tilde{\rho}_l^{(A)} = \sum_k |p_k\rangle w^0_{k-(2m+1)n_l}\langle p_k|$$

This means that the quantum object is in the state $E_l\rho_S E_l/w_l'$. Due to the normalization condition, w_l' coincides with the probability $w_l = \mathrm{Tr}_S\, E_l\rho_S$ entering (3). So the transformation (3) of the object state takes place.

APPENDIX. EXPLANATION OF VIOLATION OF THE HEISENBERG UNCERTAINTY RELATION IN OUR CASE

The Heisenberg uncertainty relation $\sigma_q\sigma_p \geq \hbar/2$ is the consequence of the well-known operator inequality

$$4\langle A^2\rangle\langle B^2\rangle \geq (i\langle[A, B]\rangle)^2 \tag{A.1}$$

valid for any self-adjoint operators A and B. Of course, it should be valid in our case.

Let us map our coordinate space onto the real axis in such a way that all points $x_n = x_0 + nL$ are images of the same coordinate-space point. Here n is an arbitrary integer. All functions on the coordinate space should appear as periodic functions on the real axis. The momentum operator $\hat{p} = -i\hbar\,\partial/\partial x$ generates shifts

$$\exp(ic\hat{p})\,\varphi(x) = \exp(c\hbar\partial/\partial x)\,\varphi(x) = \varphi(x + \hbar c)$$

in the real axis and coordinate space. The normalized eigenfunctions of $\hat{p}$ have the form

$$\varphi_k(x) = L^{-1/2}\exp(ip_k x/\hbar) \tag{A.2}$$

They correspond to eigenvalues $p_k = 2\pi\hbar k/L$.

Now the question arises of how to define the function $q(M)$ in the coordinate space (M is its point), or, which is equivalent, the function $q(x)$. We cannot set $q(x) = x$, because $q(x)$ should be periodic. However, we should define $q(x)$ in such a way that the formula

$$\psi_k(q) \overset{\text{def}}{=} L^{-1/2}\exp[ip_k x(q)/\hbar] = L^{-1/2}\exp(ip_k q/\hbar)$$

which is analogous to (A.2), is valid. For this to be so, $q(x)$ should only differ from x by periodic jumps of magnitude Δx, multiples of L, at some points $c_n = c_0 + nL$. If $0 < c \leq L/2$, we may set

$$q(x) = x - L\eta(x - c) \qquad \text{at} \quad -L/2 < x \leq L/2 \tag{A.3}$$

with $\eta(\xi) = (1 + \mathrm{sign}\,\xi)/2$. For the function (A.3) and $\hat{p} = -i\hbar\,\partial/\partial x$ we get

$$[\hat{p}, \hat{q}] = -i\hbar + i\hbar L\delta(x - c) \qquad \text{at} \quad -L/2 < x \leq L/2 \tag{A.4}$$

Averaging (A.4) or, to be exact, the matrix

$$[\hat{p}, \hat{q}]_{xx'} = -i\hbar[1 - L\delta(x - c)]\delta(x - x') \tag{A.5}$$

with density matrix $\rho_{x'x}$ of the type (22), we obtain $\langle i[\hat{p}, \hat{q}]\rangle = 0$. Therefore inequality (A.1) for $A = \hat{q}$, $B = \hat{p}$ gives $\sigma_q\sigma_p \geq 0$. So the Heisenberg uncertainty relations may be violated in our case.

The operator (A.5) in the momentum representation is of the form

$$\langle p_k | [\hat{p}, \hat{q}] | p_l \rangle = -i\hbar\delta_{kl} + i\hbar(-1)^{k-1} \tag{A.6}$$

in the limit $c \to L/2$. Therefore $\langle p_k | [\hat{p}, \hat{q}] | p_k \rangle = 0$.

Note that the unusual commutation relation (A.5), (A.6) leads to unusual dynamic equations. For example, in the case of an isolated apparatus with simple Hamiltonian $H_A = \hat{p}^2/(2m_0)$ the usual equation $\dot{\hat{q}} = \hat{p}/m_0$ is not valid.

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